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A Homoclinic Orbit in a Planar Singular ODE—A Computer Assisted Proof*

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Abstract. We consider a family of 2-dimensional ODEs of the form $\Delta_\xi(z)z' = f_\xi(z)$ depending on a real parameter ξ which was investigated by Vladimirov [*Rep. Math. Phys.*, 61 (2008), pp. 381–400]. In this system, there exist stationary points p_ξ which belong to the set of zeros of Δ_ξ. We prove, using rigorous numerics, the existence of a homoclinic orbit to p_ξ for some parameter value $\xi = \xi_h$. Due to the singularity of the system it takes a finite time to travel along this orbit, and this property gives rise to a compacton-like traveling wave in some hydrodynamic system describing relaxing media. Our approach could be used to prove similar results in other singular systems as well.

Key words. homoclinic connections, computer assisted proofs, compacton traveling waves, invariant manifolds, cone conditions

AMS subject classifications. 34C37, 37G20, 65G20

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1. Introduction.

We consider a family of ODEs of the following form:

$$\begin{cases} \Delta(x)x' = x(\sigma xy - \kappa + \tau\xi\gamma x), \\ \Delta(x)y' = -\xi(\xi x(xy - \kappa) + \chi(y + \gamma)), \end{cases} \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$ are unknown functions, $\Delta : \mathbb{R} \to \mathbb{R}$ is given by $\Delta(x) = \tau(\xi x)^2 - \chi$, and real numbers ξ, τ, κ, χ, γ, and $\sigma = 1 + \tau\xi \in \mathbb{R}$ are some physical parameters. Parameters τ, σ, κ, and γ come from the original PDE describing some relaxing media (for more details see [V] and Appendix A). The system (1) is the result of the traveling wave solution ansatz with velocity ξ. Vladimirov was particularly interested in the existence of the compacton-like solution—a traveling wave with compact support. For this he needed a "singular" fixed point for the system (1) with a homoclinic loop such that it takes finite time to travel this loop—the meaning of this statement will be explained later.

In his numerical investigations Vladimirov fixed parameters τ, κ, χ, and γ as follows:

$$\tau = 0.05, \quad \kappa = 1, \quad \chi = 4, \quad \gamma = -1, \tag{2}$$

and he looked numerically for creation of a homoclinic loop as ξ changes.

Let us discuss now some properties of (1). First of all, this ODE does not induce a local dynamical system. Zeros of $\Delta(x)$, two lines given by $x = \pm\sqrt{\frac{\chi}{\tau\xi^2}}$, introduce singularities into our system.

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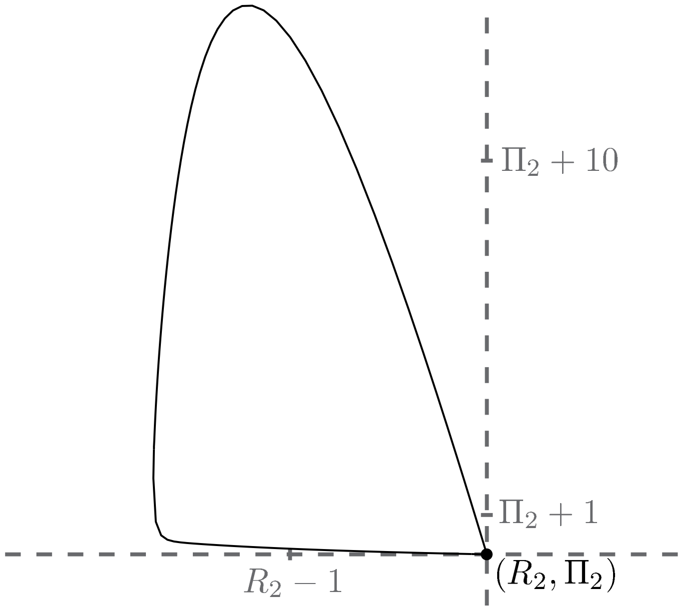

Figure 1. *A numerical approximation of the homoclinic loop in the system* (1) *with parameter values given by* (2).

System (1) has three stationary points (zeros of the right-hand side (rhs) of (1)), from which only two are interesting from the physical point of view (see [V]). The two interesting points are (R_1, Π_1) and (R_2, Π_2), where

$$R_1 = -\tfrac{\kappa}{\gamma}, \qquad \Pi_1 = -\gamma,$$
$$R_2 = \sqrt{\tfrac{\chi}{\tau\xi^2}}, \qquad \Pi_2 = \tfrac{\kappa - \tau\xi\gamma R_2}{\sigma R_2}.$$

Observe that R_2 belongs to the zero set of $\Delta(x)$—in that sense this is a singular fixed point for (1).

In [V], it was shown that for values above the critical value $\xi_{cr} = -\frac{\chi + \sqrt{\chi^2 + 4\kappa R_1^2}}{2R_1^2}$ there exists a limit cycle in the system (as a result of the Andronov–Hopf bifurcation). Good numerical evidence was presented that there exists a value $\xi_h > \xi_{cr}$ for which a homoclinic orbit appears (see Figure 1).

Our goal is to prove that there exists a (locally unique) homoclinic orbit to the point (R_2, Π_2) for some $\xi_h > \xi_{cr}$. Since we are dealing here with a singular fixed point, we are going to use a slightly modified definition of a homoclinic solution. By a homoclinic orbit to (R_2, Π_2) we understand a solution of (1) $(x, y) : (a, b) \to \mathbb{R}^2$ such that $a, b \in \mathbb{R}$, $-R_2 < x(t) < R_2$ for $t \in (a, b)$, and

$$\lim_{t\to a}(x(t), y(t)) = \lim_{t\to b}(x(t), y(t)) = (R_2, \Pi_2). \tag{3}$$

Observe that condition $|x((a, b))| < R_2$ implies that the homoclinic solution does not intersect the set of singular points of the system (1). Finite values of a and b are possible, because

(R_2, Π_2) belongs to the set $\{(x, y) : \Delta(x) = 0\}$, and this feature is essential for the construction of the compacton-like solution in [V].

The main result in this paper is the following.

Theorem 1. *Let* $\Xi = [\xi_{lo}, \xi_{up}]$, *where* $\xi_{lo} = -4.049882477$, $\xi_{up} = -4.049880977$.

For parameter values as in (2) *there exists a unique* $\xi_h \in \Xi$ *such that a homoclinic loop in the system* (1) *exists.*

Moreover, travel time along this homoclinic loop is finite.

The proof of the above theorem is a direct consequence of Theorem 10 about the existence of a homoclinic loop, Theorem 12 about the uniqueness of the loop, and Theorem 2 about the finite travel time along the homoclinic loop.

Conceptually, our method is standard (see, for example, [GH, sect. 6.1]), and its history can be traced back to Poincaré. We investigate the distance function between the stable and unstable manifolds of the hyperbolic fixed point on a suitable section of the vector field under consideration. The proof requires estimates of the stable and the unstable manifolds, together with bounds on their derivatives with respect to the parameter. Our approach is based on topological and geometrical methods developed in [ZCC]. This approach has been used by the second author and his coworkers in computer assisted proofs in dynamics of maps and ODEs [KWZ, WZ1]. These methods depend on the existence of the hyperbolic fixed point in the system under consideration. This is not the case for the point (R_2, Π_2) for the system (1). This obstacle is removed by observing that trajectories of the system (1) coincide with trajectories of the system obtained from (1) after setting $\Delta(x) = 1$ as long as singular lines are not crossed.

The method is not restricted to polynomials, and it is also possible to extend it to higher dimensions, but the topological part becomes more involved (see, for example, [WZ1], where the intersection of two 2-dimensional invariant manifolds in four dimensions was established).

Let us mention here that there is a vast literature on numerical techniques for approximating equilibria, periodic orbits, connecting orbits, and, more generally, invariant manifolds of maps or ODEs. In particular, there is a strong interest in developing computational (nonrigorous) methods for connecting orbits [B, ChK, DF, FD1, FD2, KW]. A number of authors have also developed methods that involve a combination of interval arithmetic with analytical and topological tools and provided proofs of the existence of homoclinic and heteroclinic solutions of differential equations [AAK, BL, BH, H, O, W, W1, WZ2].

Compared to these works the novelty in our approach consists of two main points. First, it allows us to prove the existence of homoclinic orbits in singular ODEs, which may arise in many applications. Second, in terms of obtained results, it allows us to prove the local uniqueness of the homoclinic loop. In fact, in all previously cited papers (except [AAK, W1]) the homo- and heteroclinic connections have been transversal, which makes the problem considerably easier.

Papers [AAK, W1], in terms of the obtained results, are very similar to our work. The authors established the existence of the traveling wave, which is obtained as the homoclinic loop to a fixed point in a 3-dimensional system with the parameter being the wave speed. In [W1], a countable set of such parameters was proven to exist, while the authors of [AAK] gave very precise estimates for the value of the wave speed—the diameter of the interval containing the homoclinic parameter is 2^{-42}. However, the authors have not tried to prove the local uniqueness of the wave speed, but there is no doubt that this can be achieved using the parameterization method they employed. The tools used in [W1] are topological in nature and exploited the symmetries of the system.

Some recent papers [AAK, BL, BH] on computer assisted proofs of the existence of the connecting orbits in flows used the parameterization method developed in [CFL1, CFL2, CFL3] to estimate rigorously the (un)stable manifolds of fixed points, for both maps and flows. This approach allows one to validate a high order expansion of the (un)stable manifold of the fixed point. Especially in the case of flows, this can yield very accurate bounds. However, the parameterization method will possibly be useless in the context of Poincaré maps for flows, taking into account the enormous computational cost of obtaining the Taylor coefficients of such maps. One way to overcome this is to parameterize the whole (un)stable manifold of the entire orbit. A construction of linear bundles for some 3-dimensional examples has been accomplished by Castelli and Lessard in [CL].

Another approach to the rigorous estimation of invariant manifolds of fixed points of ODEs has been developed by Tucker in [T1, T2, JT]. He defined a robust normal form concept, which was essential in his proof of the existence of the Lorenz attractor.

The content of this paper may be briefly described as follows: in section 2 we give an outline of the method of proving the existence and local uniqueness of a homoclinic loop in the singular system (1). In section 3 we prove that it takes finite time to travel along this homoclinic loop. In section 4 we recall from [ZCC] all necessary definitions and geometric theorems about invariant manifolds of fixed points. In section 5 we present some numerical details from our computer assisted proof.

Our program is implemented in *C++* and uses the *CAPD* library for rigorous computations [CAPD]. The source code of the program may be found on [WWW].

On the standard PC-type machine with a 2.4GHz Intel Pentium $i5$ processor the time of computation was approximately 1 minute.

1.1. Notation. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ we denote the sets of natural, integer, rational, and real numbers. For $\mathbb{R}^n$ we denote by $||x||$ the norm of x, and if the formula for the norm is not specified in some context, then any norm can be used.

Let $z_0 \in \mathbb{R}^n$; then $B_n(z_0, r) = \{z \in \mathbb{R}^n : ||z_0 - z|| < r\}$ and $B_n = B_n(0, 1)$. For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will often write $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_1(z) = \pi_x(z) = x(z) = x$ and $\pi_2(z) = \pi_y(z) = y(z) = y$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $Sp(A)$ we denote the spectrum of A. For a matrix $A \in \mathbb{R}^{n\times m}$ by A^T we denote its transpose.

For a set $U \in \mathbb{R}^n$ we denote by $\overline{U}$, $\operatorname{int} U$, and ∂U the closure, the interior, and the boundary of U, respectively.

In section 3 we will use some of the notion introduced later in section 4. If the reader is not familiar with the following terms, we provide here links to appropriate definitions. The notion of an h-set is given in Definition 1. For a given vector field f, an h-set N, and a hyperbolic fixed point $x_0 \in N$ of f we denote by $W^s_N(x_0, f)$ and $W^u_N(x_0, f)$ the stable and unstable invariant manifolds of x_0 inside the set N. Hyperbolic fixed points and stable and unstable manifolds are introduced in Definition 8.

2. Outline of the method. The basic idea of our approach to prove the existence of a homoclinic loop in (1) which is traveled in a finite time can be described as follows:

1. We desingularize our system by considering the system (1) with $\Delta(x) \equiv 1$. We obtain the following system:

$$\begin{cases} x' = x(\sigma x y - \kappa + \tau\xi\gamma x), \\ y' = -\xi(\xi x(xy - \kappa) + \chi(y + \gamma)). \end{cases} \tag{4}$$

   From now on we will denote the rhs of (4) by f_ξ. Often we will drop the subscript ξ if it is known from the context.
   It is easy to see that any solution of (4) $(x, y) : (a, b) \to \mathbb{R}^2$ such that $\Delta(x(t)) \neq 0$ for $t \in (a, b)$ gives rise to some solution of (1) and vice versa (after a suitable reparameterization of time). The direction of time is preserved if $\Delta(x(t)) > 0$ for $t \in (a, b)$ and reversed in the other case.
2. For (4) we prove that the fixed point (R_2, Π_2) is hyperbolic for all $\xi \in [\xi_{lo}, \xi_{up}]$, where ξ_{lo} and ξ_{up} are some (guessed) lower and upper estimates for ξ_h. Next, we prove that for unique $\xi_h \in [\xi_{lo}, \xi_{up}]$ left branches of the stable and unstable manifolds of the point (R_2, Π_2) coincide, giving rise to the homoclinic loop. Moreover, this loop (without the fixed point itself) is contained in the vertical strip $\{(x, y) : |x| < R_2\}$.
3. Using the fact that the homoclinic loop found in step 2 is contained in the region where $\Delta(x) < 0$ we argue that it gives rise to a solution for (1) $(x, y) : (a, b) \to \mathbb{R}^2$ such that $a, b \in \mathbb{R}$, $\Delta(x(t)) < 0$, and

$$\lim_{t\to a}(x(t), y(t)) = \lim_{t\to b}(x(t), y(t)) = (R_2, \Pi_2). \tag{5}$$

The second step takes most of the remaining part of the paper and combines abstract mathematical results with rigorous numerics. Its realization requires the following:

- Rigorous estimates of the local stable and unstable manifolds of the hyperbolic fixed point, together with their derivatives with respect to the parameter. The relevant abstract theorems will be given in section 4. Implementation details of the method are described in Lemmas 7, 8, 9 and their proofs. Lemmas 7 and 8 together with Theorem 4 give us the existence of branches of stable and unstable manifolds in the vicinity of the hyperbolic stationary point that do not cross the line of singular points. Thus we are able to integrate them outside the neighborhood of the stationary point up to some section L. Investigating the behavior on the section in Theorem 10 allows us to conclude that a homoclinic orbit exists. Lemma 9 allows us to prove the uniqueness of the homoclinic solution, and it is used in Lemma 11 and Theorem 12.
- Rigorous numerics to propagate in time the stable and unstable manifolds, together with computation of Jacobian matrices of relevant Poincaré maps. This is not discussed in this paper—the interested reader should consult [CAPD, Z]. The rigorous integration is used in the proof of Theorem 10.

The third step is realized in section 3 by analytical arguments, and the final check for nonzero coordinates of the eigenvectors in the system (4) is given in Theorem 15.

3. Finite time to travel along the homoclinic loop in (1). In this section we show that the homoclinic loop to a hyperbolic fixed point in (4) gives rise to a solution satisfying condition (3) for (1); that is, the travel time along the homoclinic orbit is finite.

Let $h : (-\infty, \infty) \to \mathbb{R}^2$ be a homoclinic solution to the point (R_2, Π_2) in (4) for $\xi = \xi_h$, and let L be a Poincaré section for (4) such that $h(0) \in L$. Moreover, assume that $\pi_x(h(t)) < R_2$ for all $t \in \mathbb{R} \setminus \{0\}$. That means $h(t)$ is a homoclinic loop for (1), because the factor $\Delta(x)$ introduces only a reparameterization of time for $h(t)$ in system (1). In view of this observation it is enough to show that once $h(t)$ is close to (R_2, Π_2) it takes finite time in the system (1) to reach (R_2, Π_2).

Theorem 2. *Let $\Delta : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $\Delta(x) = \gamma x + O(x^2)$, with $\gamma > 0$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-function.*

Consider a singular ODE

$$\Delta(x) \cdot z' = f(z), \qquad z \in \mathbb{R}^2, \tag{6}$$

and its regular modification

$$z' = f(z), \qquad z \in \mathbb{R}^2. \tag{7}$$

Assume that the following conditions are satisfied:

1. *$z_0 = (0, 0)$ is a hyperbolic fixed point of (7).*
2. *The unstable vector of $Df(z_0)$, v_u, and the stable vector of $Df(z_0)$, v_s, have a nonzero first coordinate.*

Then for any $\epsilon > 0$ there exists an h-set $N \subset B(z_0, \epsilon)$ such that N is an isolating block for (7) and the following assertions are valid:

1. *If $z : [a, b) \to \mathbb{R}^2$, where $b \in \mathbb{R} \cup \{\infty\}$, is any solution to (6) such that*
$$z(a) \in B(0, \epsilon), \quad \Delta(x(t)) > 0 \quad \forall t \in [a, b), \quad \lim_{t \to b} z(t) = z_0, \tag{8}$$
then there exists $t_0 \in [a, b)$ such that
$$z(t) \in W^s_N(z_0, f) \cap \{(x, y) \mid \Delta(x) > 0\} \quad \forall t \in [t_0, b). \tag{9}$$
2. *For any $p \in W^s_N(z_0, f) \cap \{(x, y) \mid \Delta(x) > 0\}$ there exists $b \in \mathbb{R}_+ \setminus \{0\}$ such that the forward solution of (6), denoted by $z(t)$, with an initial condition $z(a) = p$ is defined on the segment $[a, a + b)$ and the following conditions hold:*
$$z(t) \in W^s_N(z_0, f) \cap \{(x, y) \mid \Delta(x) > 0\}, \qquad t \in [a, a + b), \tag{10}$$
$$\lim_{t \to a+b} z(t) = z_0. \tag{11}$$
3. *If $z : (a, b] \to \mathbb{R}^2$, where $a \in \mathbb{R} \cup \{-\infty\}$, is any solution of (6) such that*
$$z(b) \in B(0, \epsilon), \quad \Delta(x(t)) > 0 \quad \forall t \in (a, b], \quad \lim_{t \to a} z(t) = z_0, \tag{12}$$
then there exists $t_0 \in (a, b]$ such that
$$z(t) \in W^u_N(z_0, f) \cap \{(x, y) \mid \Delta(x) > 0\} \quad \forall t \in (a, t_0]. \tag{13}$$

4. *For any* $p \in W^u_N(z_0, f) \cap \{(x,y) \mid \Delta(x) > 0\}$ *there exists* $b \in \mathbb{R}_+ \setminus \{0\}$ *such that the backward solution of* (6)*, denoted by* $z(t)$*, with an initial condition* $z(a) = p$ *is defined on the segment* $(a-b, a]$ *and the following conditions hold:*

$$z(t) \in W^u_N(z_0, f) \cap \{(x,y) \mid \Delta(x) > 0\}, \qquad t \in (a-b, a], \tag{14}$$

$$\lim_{t \to a-b} z(t) = z_0. \tag{15}$$

Proof. We prove assertions 1 and 2 only. The proofs of the other two are analogous—one needs to consider backward orbits.

Without any loss of generality we can assume that $v_s = (1, \alpha)^T$. Let $\lambda_s < 0$ be the corresponding eigenvalue. It follows that if

$$Df(z_0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{16}$$

then

$$a_{11} + a_{12}\alpha = \lambda_s. \tag{17}$$

From Theorem 3 it follows that in any neighborhood of z_0 there exists an h-set N such that N is an isolating block for (7) and $W^s_N(z_0, f)$ is contained in the cone $C = \{z_0 + t(v_s + [-\delta, \delta](0,1)^T),\ t \in \mathbb{R}\}$, where $t \in \mathbb{R}$ and δ can be chosen to be arbitrarily small. Let us choose $\delta > 0$ and $\epsilon > 0$ so small that the set defined by

$$C_+ = \{z_0 + t(v_s + [-\delta, \delta](0,1)^T) \ |t > 0\} \tag{18}$$

is contained in $B(z_0, \epsilon) \subset \{(x,y) \mid \Delta(x) > 0\}$. Now, in the domain where $\Delta(x) > 0$, the trajectories of (6) and (7) are the same up to reparameterization, and hence assertion 1 is proved.

To prove assertion 2 let us take a solution $z(t)$ of (6) with $z(a) \in W^s_N(z_0, f) \cap \{(x,y) \mid \Delta(x) > 0\}$. Then for $t \in [a, b)$ we have

$$z(t) \in W^s_N(z_0, f) \subset C_+. \tag{19}$$

Now we will study (6) in C_+.

In C_+ we have the estimate

$$x > 0, \qquad (\alpha - \delta)x \le y \le (\alpha + \delta)x. \tag{20}$$

From (6) and (17) we obtain for $(x,y) \in C_+$

$$\begin{aligned}(\gamma x + O(x^2))x' &= a_{11}x + a_{12}y + O(\|(x,y)\|^2) \\ &\in x(a_{11} + \alpha a_{12} + [-\delta, \delta]) + O(x^2) \\ &= x(\lambda_s + [-\delta, \delta]) + O(x^2).\end{aligned} \tag{21}$$

We may further decrease ϵ to ensure that for $z \in B(0, \epsilon)$ both $O(x^2)$ functions in (21) satisfy the condition

$$\|O(x^2)\| < \beta|x| \tag{22}$$

for an arbitrarily small β to be fixed later.

From (21) and (22) we obtain for $z \in C_+$ the inequality

$$(\gamma x + O(x^2))x' < x(\lambda_s + \delta + \beta),$$

and finally we get an upper bound on x':

$$x' < \frac{x(\lambda_s + \delta + \beta)}{\gamma x + O(x^2)} \leq \frac{x(\lambda_s + \delta + \beta)}{x(\gamma - \beta)} = \frac{\lambda_s + \delta + \beta}{\gamma - \beta}. \tag{23}$$

Since $\lambda_s < 0$, $\gamma > 0$ and all other constants can be made arbitrarily small, we see that we can find δ and ϵ such that

$$x' < c < 0, \qquad z \in C_+. \tag{24}$$

This means that any forward solution on $W^s_N(z_0, f) \cap \{(x, y) \mid \Delta(x) > 0\}$ will reach z_0 in finite time. ■

4. Geometric tools for invariant manifolds of fixed points.

To make this paper reasonably self-contained in this section we gather all necessary definitions and geometric theorems from [ZGi, ZCC, WZ1] related to the rigorous investigation of invariant manifolds of fixed points.

4.1. Horizontal and vertical disks.

Definition 1 (see [ZCC, Def. 1]). *An h-set N is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- *$|N|$ is a compact subset of $\mathbb{R}^n$,*
- *$u(N), s(N) \in \{0, 1, 2, \dots\}$ are such that $u(N) + s(N) = n$, and*
- *$c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that $c_N(|N|) = \overline{B}_{u(N)} \times \overline{B}_{s(N)}$.*

We set

$$\begin{aligned}
\dim(N) &:= n,\\
N_c &:= \overline{B}_{u(N)} \times \overline{B}_{s(N)},\\
N_c^- &:= \partial B_{u(N)} \times \overline{B}_{s(N)},\\
N_c^+ &:= \overline{B}_{u(N)} \times \partial B_{s(N)},\\
N^- &:= c_N^{-1}(N_c^-),\\
N^+ &:= c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an h-set N is a product of two closed balls in some coordinate system c_N. We call numbers $u(N)$ and $s(N)$ unstable and stable dimensions, respectively. The subscript c refers to the new coordinates given by c_N. The set $|N|$ is called the support of an h-set. We often drop the bars in the symbol $|N|$ and use N to denote both the h-set and its support.

Occasionally we will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_2) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an h-set. By this we will understand a "natural" h-set structure on N given by $u(N) = u$, $s(N) = s$, $c_N(x, y) = (\frac{x - x_0}{r_1}, \frac{y - y_0}{r_2})$. In the context of $\mathbb{R}^2$ and $u = 1$, $s = 1$ we will sometimes write $N = z_0 + [-a, a] \times [-b, b]$. This is compatible with the above convention, as a defines the radius of the ball $\overline{B}_u(0, a) = [-a, a]$ and b of $\overline{B}_s(0, b) = [-b, b]$.

Definition 2 (see [ZCC, Def. 5]). *Let N be an h-set. Let $b : \overline{B}_{u(N)} \to |N|$ be a continuous mapping, and let $b_c = c_N \circ b$. We say that b is* a horizontal disk *in N if there exists a homotopy*

$h : [0,1] \times \overline{B}_{u(N)} \to N_c$ *such that*

$$h_0 = b_c, \tag{25}$$

$$h_1(x) = (x,0) \quad \forall x \in \overline{B}_{u(N)}, \tag{26}$$

$$h(t,x) \in N_c^- \quad \forall t \in [0,1] \text{ and } \forall x \in \partial B_{u(N)}. \tag{27}$$

Definition 3 (see [ZCC, Def. 6]). *Let N be an h-set. Let $b : \overline{B}_{s(N)} \to |N|$ be a continuous mapping, and let $b_c = c_N \circ b$. We say that b is* a vertical disk *in N if there exists a homotopy $h : [0,1] \times \overline{B}_{s(N)} \to N_c$ such that*

$$h_0 = b_c, \tag{28}$$

$$h_1(x) = (0,x) \quad \forall x \in \overline{B}_{s(N)}, \tag{29}$$

$$h(t,x) \in N_c^+ \quad \forall t \in [0,1] \text{ and } \forall x \in \partial B_{s(N)}. \tag{30}$$

Definition 4 (see [ZCC, Def. 7]). *Let N be an h-set in $\mathbb{R}^n$, and let b be a* horizontal *(*vertical*)* disk *in N. We will say that $x \in \mathbb{R}^n$* belongs *to b when $b(z) = x$ for some $z \in$* dom(b).

By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs *to b.*

4.2. Cone conditions and the stable manifold theorem. Below we recall definitions and theorems that allow us to handle and verify hyperbolic structures of ODEs using h-sets and quadratic forms.

Definition 5 (see [ZCC, Def. 8]). *Let $N \subset \mathbb{R}^n$ be an h-set, and let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form such that*

$$Q(x,y) = \alpha(x) - \beta(y), \qquad (x,y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}, \tag{31}$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$ and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N,Q) is called an h-set with cones.

We will often omit Q in the symbol (N,Q) and will say that N is an h-set with cones.

Definition 6 (see [ZCC, Def. 9]). *Let (N,Q) be an h-set* with cones *and $b : \overline{B}_u \to |N|$ be a horizontal disk.*

We will say that b satisfies the cone condition (with respect to Q) *iff for any $x_1, x_2 \in \overline{B}_u$, $x_1 \neq x_2$ the following inequality holds:*

$$Q(b_c(x_1) - b_c(x_2)) > 0. \tag{32}$$

Definition 7 (see [ZCC, Def. 10]). *Let (N,Q) be an h-set* with cones *and $b : \overline{B}_s \to |N|$ be a vertical disk.*

We will say that b satisfies the cone condition (with respect to Q) *iff for any $y_1, y_2 \in \overline{B}_s$, $y_1 \neq y_2$ the following inequality holds:*

$$Q(b_c(y_1) - b_c(y_2)) < 0. \tag{33}$$

Let us consider an ODE

$$z' = f(z), \tag{34}$$

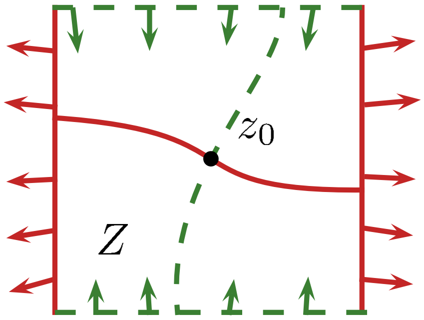

Figure 2. *An isolating block Z for some planar ODE. The stable (vertical green dashed line) and unstable (horizontal red solid line) manifolds for φ inside Z are plotted; arrows indicate the vector field f. Dashed green and solid red border lines indicate the δ-sections Σ^+ and Σ^-, respectively.*

where $z \in \mathbb{R}^n$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

Let us denote by $\varphi(t,p)$ the solution of (34) with the initial condition $z(0) = p$.

The following definition is standard.

Definition 8. *Let $z_0 \in \mathbb{R}^n$. We say that z_0 is a hyperbolic fixed point for (34) iff $f(z_0) = 0$ and $Re\lambda \neq 0$ for all $\lambda \in Sp(Df(z_0))$, where $Df(z_0)$ is the derivative of f at z_0.*

Let $z_0 \in Z \subset \mathbb{R}^n$ be a hyperbolic fixed point for (34). We define

$$W^s_Z(z_0, \varphi) = W^s_Z(z_0, f) = \left\{ z : \forall_{t \geq 0} \varphi(t,z) \in Z, \quad \lim_{t \to \infty} \varphi(t,z) = z_0 \right\}, \tag{35}$$

$$W^u_Z(z_0, \varphi) = W^u_Z(z_0, f) = \left\{ z : \forall_{t \leq 0} \varphi(t,z) \in Z, \quad \lim_{t \to -\infty} \varphi(t,z) = z_0 \right\}. \tag{36}$$

Sometimes, when it is known from the context, φ will be dropped and we will write $W^s_Z(z_0)$, etc. $W^s_Z(z_0, \varphi)$ is called the stable manifold for φ (or for f) in Z, and $W^u_Z(z_0, \varphi)$ is called the unstable manifold for φ in Z. Geometric interpretation may be found in Figure 2.

Below we recall the notion of the isolating block from the Conley index theory.

Definition 9. *For $\delta > 0$ the set $\Sigma \subset \mathbb{R}^n$ is called a* δ-section *for the flow φ iff $\varphi((-\delta,\delta), \Sigma)$ is an open set and the map $\sigma : \Sigma \times (-\delta, \delta) \to \varphi((-\delta,\delta),\Sigma)$ defined by $\sigma(x,t) = \varphi(t,x)$ is a homeomorphism.*

Let $B \subset \mathbb{R}^n$ be a compact set. B is called an isolating block *iff $\partial B = B^- \cup B^+$, where B^- and B^+ are closed sets, and there exist $\delta > 0$ and two δ-sections, Σ^+ and Σ^-, such that*

$$\begin{aligned} B^+, \subset \Sigma^+, \quad & B^- \subset \Sigma^-, \\ \forall x \in B^+, \quad \forall t \in (-\delta, 0) \qquad & \varphi(t,x) \notin B, \\ \forall x \in B^-, \quad \forall t \in (0, \delta) \qquad & \varphi(t,x) \notin B. \end{aligned}$$

In the present paper we will use h-sets which are isolating blocks. Simply, it means that N^+ and N^- are sections of the vector field.

Definition 10. *Let N be an h-*set *in $\mathbb{R}^n$. We say that N is an isolating block for ODE (34) iff N^- and N^+ are the δ-sections for f as in Definition 9.*

Definition 11. *Let N be an h-*set *such that c_N is a diffeomorphism. For a vector field f on $|N|$ we define a vector field on N_c by*

$$f_c(z) = Dc_N(c_N^{-1}(z)) f(c_N^{-1}(z)). \tag{37}$$

Observe that f_c is in fact the vector field f expressed in the new variables.

Definition 12 (see [ZCC, Def. 13]). *Let $U \subset \mathbb{R}^n$ be such that $U = \overline{U}$ and $\operatorname{int} U \neq \emptyset$. Let $g : U \to \mathbb{R}^m$ be a C^1 function. We define the interval enclosure of $Dg(U)$ by*

$$[Dg(U)] := \left\{ A \in \mathbb{R}^{n\times m} : \forall_{i,j} A_{ij} \in \left[\inf_{x\in U} \frac{\partial g_i}{\partial x_j}, \sup_{x\in U} \frac{\partial g_i}{\partial x_j} \right] \right\}. \tag{38}$$

We say that $[Dg(U)]$ is positive definite if for all $A \in [Dg(U)]$ the matrix A is positive definite.

The following two theorems about existence and local properties of the (un)stable manifold of the hyperbolic fixed point follow immediately from the proof of Theorem 26 in [ZCC].

Theorem 3. *Let $n = u+s$, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-function with z_0 a hyperbolic fixed point for f such that*

$$Df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix}, \tag{39}$$

where $A \in \mathbb{R}^{u\times u}$, $U \in \mathbb{R}^{s\times s}$ such that $A + A^T$ is positive definite and $U + U^T$ is negative definite.

Then for any $\epsilon > 0$ and for any quadratic form $Q(x, y) = ax^2 - by^2$, $x \in \mathbb{R}^u$, $y \in \mathbb{R}^s$, $a > 0$, $b > 0$, there exists an h-set *$N = z_0 + \overline{B}_u(0, r) \times \overline{B}_s(0, r) \subset B(z_0, \epsilon)$ such that N is an isolating block for $x' = f(x)$, $W^s_N(z_0, f)$ is a vertical disk in N satisfying the cone condition, and $W^u_N(z_0, f)$ is a horizontal disk in N satisfying the cone condition.*

Theorem 4. *Assume that (N, Q) is an* h-set with cones, *which is an isolating block for (34), c_N is a diffeomorphism, and the following cone condition is satisfied:*

$$\text{The matrix} \quad [Df_c(N_c)]^T Q + Q[Df_c(N_c)] \quad \text{is positive definite.} \tag{40}$$

Then there exists $z_0 \in N$ such that $f(z_0) = 0$, $W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition, and $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

To explain the meaning of condition (40) let us remark that it implies the following fact: assume that $z_i : [0, T] \to N$ for $i = 1, 2$ are two different orbits of our ODE; then

$$\frac{d}{dt} Q(z_1(t) - z_2(t)) > 0, \qquad t \in [0, T]. \tag{41}$$

For the justification see [ZCC] or the proof of Theorem 5 in the present paper.

4.3. Dependence of the (un)stable manifold on parameters. The following theorem gives computable bounds on the first derivatives of stable and unstable manifolds of a hyperbolic fixed point for an ODE with respect to the parameter. An analogous theorem for maps was given in [ZCC, Thm. 21] with some refined estimates in [WZ1, Thm. 4.1]. The theorem below and its proof are an adaptation of these results to the ODE setting.

We will be using the norm for the symmetric bilinear forms (identified in what follows with the symmetric matrices) which is defined by

$$|B(u, v)| \leq \|B\| \|u\| \|v\|.$$

For the Euclidean norm we have

$$\|B\| = \max\{|s| \ : \ s \text{ is an eigenvalue of } B\}.$$

Theorem 5. *Let $\Xi \subset \mathbb{R}$ be a compact interval. Let $f : \Xi \times \mathbb{R}^{u+s} \to \mathbb{R}^{u+s}$ be a C^1 function. For $\xi \in \Xi$ we consider a one-parameter family of ODEs*

$$x' = f(\xi, x) = f_\xi(x). \tag{42}$$

Assume that $Q(x, y) = \alpha(x) - \beta(y) = \sum_{i=1}^{u} a_i x_i^2 - \sum_{i=1}^{s} b_i y_i^2$, where $a_i > 0$ and $b_i > 0$ for all i. Let B be a symmetric bilinear form such that $B(z, z) = Q(z)$ for all $z \in \mathbb{R}^{u+s}$.

Assume that (N, Q) is an h-set with cones in $\mathbb{R}^{u+s}$ such that c_N is a diffeomorphism.

1. *Let $A > 0$ be such that for all $\xi \in \Xi$, for all $z \in \mathbb{R}^{u+s}$ the following inequality is true:*

$$z^T([Df_{\xi,c}(N_c)^T]Q + Q[Df_{\xi,c}(N_c)])z \geq A\|z\|^2. \tag{43}$$

2. *Assume that the h-set N is an isolating block for (42) for all $\xi \in \Xi$. Let p_ξ denote the fixed point for f_ξ, which is unique due to 1.*
3. *Let*

$$D = \max_{i=1,\dots,u} a_i, \tag{44}$$

$$L = \sup_{\xi \in \Xi, z \in N} \left\| \frac{\partial f_{\xi,c}}{\partial \xi}(z) \right\|. \tag{45}$$

4. *Let $\delta > 0$ be such that*

$$\delta < \frac{A^2}{4 \cdot \|B\|^2 \cdot L^2 D}. \tag{46}$$

Then the set $W_N^s(p_\xi, f_\xi)$ can be parameterized as a vertical disk in $N \times \Xi$ for the quadratic form $\tilde{Q}(z, \xi) = \delta Q(z) - \xi^2$. By this we mean that there exists a function $x_s : \Xi \times \overline{B}_s(0, 1) \to \overline{B}_u(0, 1)$ such that $c_N(W_N^s(p_\xi, f_\xi)) = \{(x_s(\xi, y), y) \mid y \in \overline{B}_u(0, 1)\}$ and for any pair $(\xi_i, y_i) \in \Xi \times \overline{B}_s(0, 1)$, $i = 1, 2$, such that $(\xi_1, y_1) \neq (\xi_2, y_2)$ the following inequality is true:

$$\delta\alpha(x_s(\xi_1, y_1) - x_s(\xi_2, y_2)) - \delta\beta(y_1 - y_2) - (\xi_1 - \xi_2)^2 < 0. \tag{47}$$

Proof. We will assume that $c_{|N} = \mathrm{Id}$. Hence $N_c = N$ and $f_{\xi,c} = f_\xi$.

Let us consider an extended system

$$\dot{x} = f_\xi(x), \qquad \dot{\xi} = 0 \tag{48}$$

and the set

$$S(\delta) = \{((\xi_1, z_1), (\xi_2, z_2)) \in (\Xi \times N)^2 \mid \xi_1 \neq \xi_2, \ (\xi_1 - \xi_2)^2 \leq \delta Q(z_1 - z_2)\}.$$

Lemma 6. *Let δ be as in the assumptions of our theorem. Assume that $(\xi_i, z_i(t))$ for $i = 1, 2$, $t \in [0, T]$, are solutions of (48) such that $z_i([0, T]) \subset N$ and $((\xi_1, z_1(0)), (\xi_2, z_2(0))) \in S(\delta)$.*

Then $((\xi_1, z_1(t)), (\xi_2, z_2(t))) \in S(\delta)$ *for* $t \in [0, T]$.

Moreover, we have the following estimate for some $\epsilon > 0$*, which does not depend on* T*:*

$$Q(z_1(t) - Q(z_2(t)) \geq Q(z_1(0) - z_2(0))e^{\epsilon t}, \qquad t \in [0, T]. \tag{49}$$

We will prove this lemma after we complete the current proof of Theorem 5.

Let $z_i \in W^s_N(p_{\xi_i}, f_{\xi_i})$ for $i = 1, 2$. We will argue by contradiction that $(\xi_1 - \xi_2)^2 > \delta Q(z_1 - z_2)$.

Assume the contrary. Let us consider first the case $\xi_1 \neq \xi_2$.

Then $((\xi_1, z_1), (\xi_2, z_2)) \in S(\delta)$. From Lemma 6 we obtain for $\epsilon > 0$

$$Q(z_1(t) - z_2(t)) \geq Q(z_1 - z_2)e^{\epsilon t}, \qquad t \in [0, \infty). \tag{50}$$

Since $Q(z_1 - z_2) > 0$, we obtain that $Q(z_1(t) - z_2(t))$ is unbounded. However, this is impossible, because $z_i(t) \in N$. This proves that if $\xi_1 \neq \xi_2$, then

$$(\xi_1 - \xi_2)^2 > \delta Q(z_1 - z_2). \tag{51}$$

If $\xi_1 = \xi_2$, then the above inequality follows from the fact that for any given ξ the set $W^s_N(p_\xi, f_\xi)$ is a vertical disk in N satisfying the cone conditions with respect to the quadratic form Q (see Theorem 4). ∎

Proof of Lemma 6. In the proof we will use Q to denote both the quadratic form and the symmetric matrix, i.e., $Q(z) = z^T Q z$. Analogously for symmetric bilinear form B we will use the symbol B, i.e., $B(u, w) = u^T B w$. Now if $Q(z) = B(z, z)$ for all z, then $Q = B$ as matrices.

We have

$$\begin{aligned}
&\frac{d}{dt} Q(z_1(t) - z_2(t)))_{|t=0} \\
&= (f_{\xi_1}(z_1) - f_{\xi_2}(z_2))^T Q(z_1 - z_2) + (z_1 - z_2)^T Q(f_{\xi_1}(z_1) - f_{\xi_2}(z_2)) \\
&= ((f_{\xi_1}(z_1) - f_{\xi_1}(z_2)) + (f_{\xi_1}(z_2) - f_{\xi_2}(z_2)))^T Q(z_1 - z_2) \\
&\quad + (z_1 - z_2)^T Q((f_{\xi_1}(z_1) - f_{\xi_1}(z_2)) + (f_{\xi_1}(z_2) - f_{\xi_2}(z_2)))) \\
&= (z_1 - z_2)^T C^T Q(z_1 - z_2) + (z_1 - z_2)^T Q C(z_1 - z_2) \\
&\quad + 2B(f_{\xi_1}(z_2) - f_{\xi_2}(z_2), z_1 - z_2) \\
&\geq A\|z_1 - z_2\|^2 - 2\|B\| \cdot \sup_{(z,\xi) \in N \times \xi} \left\| \frac{\partial f_\xi}{\partial \xi} \right\| \cdot |\xi_1 - \xi_2| \cdot \|z_1 - z_2\|,
\end{aligned}$$

where

$$\begin{aligned}
C = C_{\xi_1}(z_1, z_2) &= \int_0^1 df_{\xi_1}(z_1 + t(z_2 - z_1))dt \\
&\in [df_\xi(\Xi \times N)]^T Q + Q\, [df_\xi(\Xi \times N)] .
\end{aligned}$$

Observe that by the definition of D we have for $((\xi_1, z_1(0)), (\xi_2, z_2(0))) \in S(\delta)$ the following inequalities:

$$(\xi_1 - \xi_2)^2 \leq \delta Q(z_1 - z_2) \leq \delta D(z_1 - z_2)^2. \tag{52}$$

Combining the above computations altogether we obtain for $((\xi_1, z_1(0)), (\xi_2, z_2(0))) \in S(\delta)$ that

$$\begin{aligned}\frac{d}{dt}Q(z_1(t) - z_2(t))_{|t=0} &\geq A\|z_1 - z_2\|^2 - 2\|B\| \cdot L \cdot \sqrt{\delta D}(z_1 - z_2)^2 \\ &= \left(A - 2\|B\| \cdot L \cdot \sqrt{\delta D}\right)(z_1 - z_2)^2 \geq \epsilon_1 (z_1 - z_2)^2 \geq \epsilon Q(z_1 - z_2),\end{aligned}$$

where $\epsilon_1 > 0$ and $\epsilon > 0$ are such that

$$\begin{aligned}A - 2\|B\| \cdot L \cdot \sqrt{\delta D} > \epsilon_1 > 0, \\ \epsilon_1 (z_1 - z_2)^2 \geq \epsilon Q(z_1 - z_2).\end{aligned}$$

Therefore the set $S(\delta)$ is forward invariant relative to $\Xi \times N$. We have proved also that

$$\frac{dQ(z_1 - z_2)}{dt} \geq \epsilon Q(z_1 - z_2); \tag{53}$$

from this (49) follows easily. ■

5. Details of computer assisted proof of the existence and local uniqueness of the homoclinic loop in (4). To establish the existence of the homoclinic loop in system (4) we proceed in a standard way (see [GH, sect. 6.1]). We fix a section of the vector field (4) (we denote it by L) such that it is parameterized by one coordinate (we denote it by v). Let $\Xi = [\xi_{lo}, \xi_{up}]$ be as in Theorem 1. We prove that for all $\xi \in \Xi$ the unstable and stable manifolds of the hyperbolic fixed point p_ξ intersect L at points $v_u(\xi)$ and $v_s(\xi)$, respectively. Observe that the functions v_u and v_s are smooth.

In the case of the homoclinic loop we should have $v_u(\xi) = v_s(\xi)$ for some ξ. Therefore we consider a function

$$T(\xi) = v_s(\xi) - v_u(\xi). \tag{54}$$

We show that the signs of $T(\xi_{lo})$ and $T(\xi_{up})$ are opposite; hence, by the intermediate value theorem, there exists $\xi_h \in \Xi$ such that $T(\xi_h) = 0$. This gives us the homoclinic loop for $\xi = \xi_h$. To prove the uniqueness of this loop we show that $T'(\xi)$ has a constant sign.

Let us now turn to the computation of $v_u(\xi)$ and $\frac{dv_u}{d\xi}$. The point $v_u(\xi)$ is defined as the first intersection of the left branch of $W^u(p_\xi, f_\xi)$ (the branch which enters the half plane $x < R_2$) with the section L. Since we are in dimension two, the left branch of $W^u(p_\xi, f_\xi)$ is a single trajectory. To be able to estimate $v_u(\xi)$ and its derivative we use Theorems 4 and 5. Namely, we construct an h-set with cones (U, Q) satisfying assumptions of these theorems. This gives us a point $z_u(\xi) \in U^- \cap W^u_U(p_\xi, f_\xi) \cap \{(x, y) : |x| \leq R_2\}$; moreover, we also have a bound on $\frac{d}{d\xi} z_u$. Now $v_u(\xi)$ can be defined by $v_u(\xi) = P_{U_l^- \to L}(\xi, z_u(\xi))$, where $P_{U_l^- \to L}$ is the Poincaré map from the neighborhood of the section U_l^-—the left component of U^- (i.e., contained in $\{(x, y) \mid x < R_2\}$) to the section L.

We have the following formula for v_u':

$$v_u'(\xi) = \frac{\partial}{\partial \xi} P_{U_l^- \to L}(\xi, z_u(\xi)) + \frac{\partial}{\partial z} P_{U_l^- \to L}(\xi, z_u(\xi)) z_u'(\xi). \tag{55}$$

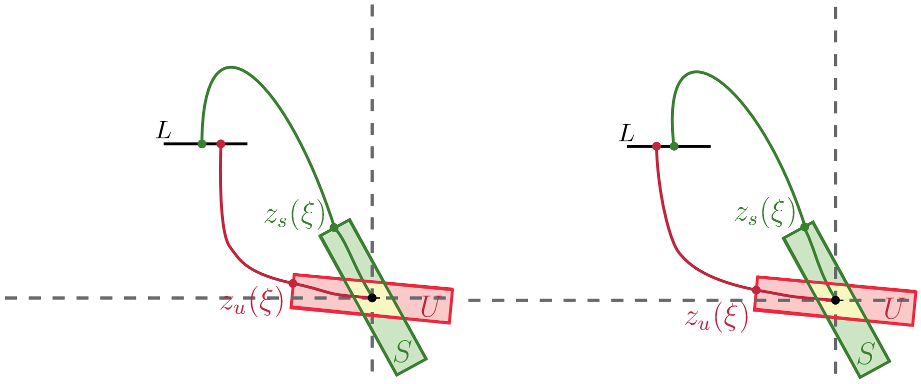

Figure 3. *A schematic figure presenting the idea of the proof of Theorem 10. The left figure presents stable and unstable branches for ξ_{lo} and the right one for ξ_{up}.*

The partial derivatives of $P_{U_l^- \to L}$ with respect to ξ and z can be computed with any rigorous integrator for ODEs; we use the CAPD library for this purpose [CAPD].

Estimates of $v_s(\xi)$ and its derivative are obtained in an analogous way. The schematic picture of the above idea of the proof is presented in Figure 3.

Warning about the presentation of the numbers in the remainder of this section. All the computations in our program have been done using interval arithmetics based on the double precision representation of the real numbers. We believe that the readability of our description of the computer assisted proof will be enhanced if we also include some numerical data from the proof. Since giving the exact values from the computer program in the binary representation will make them unreadable, we decided to write instead nonrigorous representations with 8 significant digits of rigorous numbers (intervals) produced by our program. Hence in the strict sense most of the lemmas listed in this section cannot be regarded as "proven rigorously with computer assistance," but they "reflect" their true counterparts proved by our program.

5.1. Local estimates on W^u and W^s. From now on, in this section, we will assume that Ξ is defined as in Theorem 1. All steps can be reproduced for other singular systems as long as the Δ is a polynomial in one of the coordinates and the rhs of the equation is a C^1 function. In fact our program that conducts the computer assisted proof presented below may be used to prove analogues of Theorem 1 for other values of parameters τ, κ, χ, and γ (see the documentation of the source code for details on how to do this). We have chosen to present the steps of our method on a concrete example rather than in form of an abstract algorithm, as we believe this way is better for the reader to grasp the idea of the method and to understand difficulties which can arise during computations.

For a given value of ξ we will slightly abuse the notation and denote by Δ_ξ the set

$$\Delta_\xi = \{(x, y) \in \mathbb{R}^2 : \Delta(x) = 0\}. \tag{56}$$

By $\varphi_\xi(t, z)$ we denote the solution of (4) with the initial condition given by z and the parameter ξ. If ξ is known from the context, we will omit the subscript ξ.

We want to point out that for an h-set (N, Q) that satisfies the assumptions of Theorem 4 we get information on both $W^u(p_\xi)$ and $W^s(p_\xi)$ at the same time. However, this is not optimal for finding good enclosures for stable and unstable manifolds of p_ξ. Instead we choose an h-set U, which is a parallelogram relatively large in the unstable direction and thin in the stable direction, and thus we obtain a good estimate on $W^u(p_\xi)$. Analogously, we choose an S which is short in the unstable direction and large in the stable direction to get a better estimate on $W^s(p_\xi)$.

The sets S and U should be chosen so that their sides are approximately parallel to the eigenvectors of Df computed at the stationary point p_ξ. Notice that the stationary point (R_2, Π_2) moves as the parameter ξ is changing. For this purpose we choose the middle point of the interval hull of the set of all stationary points for $\xi \in \Xi$ as a good candidate for the center of the sets S and U. We will denote this point by $\hat{z}_0$. Next, we choose coordinate maps c_S and c_U in such a way that sets S and U are aligned along the eigenvectors of $Df(\hat{z}_0)$ with a slight adjustment to c_S in order to have an isolating block reaching further along the stable manifold.

The eigenvectors and eigenvalues of p_ξ can be computed analytically. It turns out that the eigenvectors almost coincide with the coordinate axes, with OX close to the unstable direction with eigenvalue $\lambda_u \approx 0.62$ and OY being close to the stable direction with eigenvalue $\lambda_S \approx -63.6$.

Lemma 7. *We define two h-sets U and S on the plane $\mathbb{R}^2$ as follows:*

- $u(U) = s(U) = u(S) = s(S) = 1$;
- $\hat{z}_0 = (2.208526696, 0.313849028) \in \mathbb{R}^2$;
- $M_U = \begin{bmatrix} -0.9951981155 & -0.06027571515 \\ 0.09788110563 & 0.9981817661 \end{bmatrix}$;
- $M_S = \begin{bmatrix} -0.9981817661 & -0.06027571515 \\ -0.06027571515 & 0.9981817661 \end{bmatrix}$;
- $|U| = \hat{z}_0 + M_U \cdot ([-a_U, a_U] \times [-b_U, b_U])$ *with* $a_U = 0.005$, $b_U = 0.0001$, $c_U^{-1}(x, y) = \hat{z}_0 + M_U(a_U x, b_U y)^T$;
- $|S| = \hat{z}_0 + M_S \cdot ([-a_S, a_S] \times [-b_S, b_S])$ *with* $a_S = 0.00005$, $b_S = 0.01$, $c_S^{-1}(x, y) = \hat{z}_0 + M_S(a_S x, b_S y)^T$.

Then, for all $\xi \in \Xi$, the h-sets S and U are the isolating blocks for (4). Moreover, the following are true:

- $U^- \cap \Delta_\xi = \emptyset$;
- $S^+ \cap \Delta_\xi = \emptyset$.

Proof. For a given h-set N let us denote by n_z^+ the outside normal vector at a given point $z \in N^+$, and by n_z^- let us denote the outside normal vector for $z \in N^-$. It suffices to check for each $\xi \in \Xi$ that for all $z \in N^+$ the inner product $\langle f_\xi(z), n_z^+ \rangle$ is negative and for each $z \in N^-$ $\langle f_\xi(z), n_z^- \rangle$ is positive.

We checked this condition using the rigorous arithmetics and obtained for all $\xi \in \Xi$

$$
\begin{aligned}
\langle f_\xi(z), n_z^+ \rangle &\in [-0.006897320012, -0.005714477667] < 0, \quad z \in U^+, \\
\langle f_\xi(z), n_z^- \rangle &\in [0.002076688005, 0.00409304022] > 0, \quad z \in U^-, \\
\langle f_\xi(z), n_z^+ \rangle &\in [-0.6351155027, -0.633170357] < 0, \quad z \in S^+, \\
\langle f_\xi(z), n_z^- \rangle &\in [0.00001121576298, 0.00005048696143] > 0, \quad z \in S^-,
\end{aligned}
$$

which prove that S and U are the isolating blocks for (4).

For the second part we obtained that for all $\xi \in \Xi$

$$\begin{aligned}\pi_x(U^-) &\subset [2.203544678, 2.203556733] \cup [2.213496659, 2.213508714],\\ \pi_x(S^+) &\subset [2.20787403, 2.207973848] \cup [2.209079544, 2.209179362],\\ \pi_x(\Delta_\xi) &\subset [2.208526287, 2.208527105] \cup [-2.208527105, -2.208526287].\end{aligned}$$

It is easy to see that $U^- \cap \Delta_\xi = \emptyset$ and $S^+ \cap \Delta_\xi = \emptyset$. ■

In the following lemma we define Q for U and S. Observe that while the formula for Q is the same for both U and S, the cones are different, as the formula is given in the internal coordinates of U and S (defined by c_U and c_S in Lemma 7).

Lemma 8. *Assume that U and S are as in Lemma 7, and let $Q(x, y) = x^2 - y^2$.*

Let us consider (4) with parameter values as in (2). Let $p_\xi = (R_2, \Pi_2)$.

Then for all $\xi \in \Xi$ the h-sets with cones (U, Q) and (S, Q) satisfy the assumptions of Theorem 4 and there exist points $z_s(\xi) \in S^+ \cap W^s_S(p_\xi, f_\xi)$ and $z_u(\xi) \in U^+ \cap W^u_U(p_\xi, f_\xi)$ such that

$$z_s(\xi), z_u(\xi) \in \{(x, y) \in \mathbb{R}^2 : |x| < R_2\}, \tag{57}$$

$$\varphi_\xi(t, z_s(\xi)) \cap \Delta_\xi = \emptyset, \quad \varphi_\xi(-t, z_u(\xi)) \cap \Delta_\xi = \emptyset, \quad t \geq 0. \tag{58}$$

Proof. We use rigorous numerics to check that

$$\begin{aligned}&[Df_c(U_c)]^T Q + Q[Df_c(U_c)]\\ &\qquad \subset \begin{pmatrix} [1.20236854, 1.267286676] & [-6.538562255, 6.532945059] \\ [-6.538562255, 6.532945059] & [126.09537, 127.5777877] \end{pmatrix},\\ &[Df_c(S_c)]^T Q + Q[Df_c(S_c)]\\ &\qquad \subset \begin{pmatrix} [1.075080553, 1.39456723] & [-2.043226555, 1.940940435] \\ [-2.043226555, 1.940940435] & [126.653084, 127.0184426] \end{pmatrix},\end{aligned}$$

which are positive definite.

From Theorem 4 and Lemma 7 we get the existence of $z_u(\xi) \in U^- \cap W^u_U(p_\xi, f_\xi))$ and $z_s(\xi) \in S^+ \cap W^s_S(p_\xi, f_\xi)$ such that they satisfy condition (57). It remains to show (58).

For the h-set N and the point $p \in N_c$ let $K^+_{N,c}(p) = \{z \in N_c : Q(z - p) > 0\}$ and $K^-_{N,c}(p) = \{z \in N_c : Q(z - p) < 0\}$. Then we define the positive and negative cones of an h-set N at the point $q \in |N|$ by $K^\pm_N(q) = c_N^{-1}\left(K^\pm_N(c_N(q))\right)$, respectively.

By the definition of Q and by Theorem 4 we know that

- $W^s_S(p_\xi)$ lies inside the negative cone $K^-_S(p_\xi)$, and
- $W^u_U(p_\xi)$ lies inside the positive cone $K^+_U(p_\xi)$.

We want to show that for each $\xi \in \Xi$ we have the following inclusions that guarantee (58):

$$\Delta_\xi \cap S \subset \overline{K^+_S(p_\xi)}, \tag{59}$$

$$\Delta_\xi \cap U \subset \overline{K^-_U(p_\xi)}. \tag{60}$$

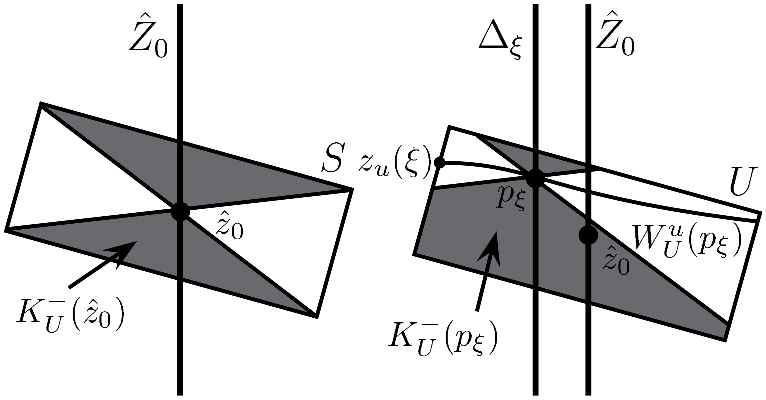

Figure 4. *A schematic figure presenting the idea of the proof of Lemma 8 for the case of h-set U (the case for S is analogous). The cones in gray present the Lipschitz dependence imposed by choosing a suitable quadratic form which ensures that W_S^s and W_U^u do not cross Δ_ξ. In the left figure the point $\hat{z}_0$ and the set $\hat{Z}_0$ are shown. In the right one the point p_ξ and the set Δ_ξ are shown.*

By Lemma 7 we know that for the set $\hat{Z}_0 = \{(x,y) : x = \hat{z}_0\}$ we have $\hat{Z}_0 \cap S \subset \overline{K_S^+(\hat{z}_0)}$ and $\hat{Z}_0 \cap U \subset \overline{K_U^-(\hat{z}_0)}$. Now (59) and (60) follow easily, as Δ_ξ and p_ξ are simply $\hat{Z}_0$ and $\hat{z}_0$ translated by the vector $p_\xi - \hat{z}_0$ (see Figure 4). ■

Lemma 9. *Assume that (S,Q), (U,Q), z_u, z_s are as in Lemma 8.*

Let $x_s(\xi) = \pi_x c_U(z_s(\xi))$ and $y_u(\xi) = \pi_y c_S(z_u(\xi))$, i.e., $z_s(\xi) = z_0 + M_S \cdot (x_s(\xi)\cdot a_S, b_S)^T = c_S(x_s(\xi),1)^T$ and $z_u(\xi) = z_0 + M_U \cdot (a_U, y_u(\xi)\cdot b_U)^T = c_U(1, y_u(\xi))^T$.

Then, the following inequalities hold true:

$$\left|\frac{\partial x_s}{\partial \xi}(\xi)\right| < \varepsilon_S = 16748.53759, \tag{61}$$

$$\left|\frac{\partial y_u}{\partial \xi}(\xi)\right| < \varepsilon_U = 67667.96072. \tag{62}$$

Proof. The proof is a consecutive computation of the quantities appearing in the assumptions of Theorem 5.

First we focus on the stable case. We checked rigorously that for $A = 1.041839776$ we have $z^T([Df_{\xi,c}(S_c)^T]Q + Q[Df_{\xi,c}(S_c)])z \geq A \cdot z^2$ for all $z \in \mathbb{R}^2$ and $\xi \in \Xi$.

Let

$$\begin{aligned} D &= 1, \\ L &= \sup_{\xi\in\Xi, z\in U_c} \left\|\frac{\partial f_{\xi,c}}{\partial \xi}(z)\right\| \in [7638.714144, 8724.646326], \\ \|B\| &= 1. \end{aligned}$$

By (47) we obtain (setting $y_1 = y_2 = 1$) for any $\xi_1, \xi_2 \in \Xi$

$$\delta_S(x_s(\xi_1) - x_s(\xi_2))^2 \leq (\xi_1 - \xi_2)^2. \tag{63}$$

Therefore for $\xi_1 \neq \xi_2$ we obtain

$$\left|\frac{x_s(\xi_1) - x_s(\xi_2)}{\xi_1 - \xi_2}\right| \leq \frac{1}{\sqrt{\delta_S}} \tag{64}$$

and finally after passing to the limit $\xi_2 \to \xi_1$ we obtain

$$\left|\frac{dx_s(\xi)}{d\xi}\right| \leq \frac{1}{\sqrt{\delta_S}}. \tag{65}$$

By the expression for δ_S we get that $\frac{1}{\sqrt{\delta_S}} \leq 16748.53759$.

To treat the unstable case we will use the fact that the unstable manifold for f becomes a stable manifold for the vector field given by $z' = -f(z)$. Of course we need to use the quadratic form $-Q$ in this case. However, the expression for the cone condition does not change:

$$z^T \left([-df(U)^T](-Q) + (-Q)[-df(U)]\right) z = z^T \left([df(U)^T]Q + Q[df(U)]\right) z.$$

Again, we check rigorously that for $A = 1.103520275$ we have $z^T([df_{\xi,c}(U_c)^T]Q+Q[df_{\xi,c}(U_c)])z \geq A \cdot z^2$ for all $z \in \mathbb{R}^2$.

We take

$$\begin{aligned}
D &= 1,\\
L &= \sup_{\xi\in\Xi, z\in U_c} \left\|\frac{\partial f_{\xi,c}}{\partial \xi}(z)\right\| \in [26670.51178, 29130.44252],\\
\|B\| &= 1.
\end{aligned}$$

Again, by (47) we obtain (setting $x_1 = x_2 = 1$)

$$\delta_U(y_u(\xi_1) - y_u(\xi_2))^2 \leq (\xi_1 - \xi_2)^2. \tag{66}$$

For $\xi_1 \neq \xi_2$ we obtain

$$\left|\frac{y_u(\xi_1) - y_u(\xi_2)}{\xi_1 - \xi_2}\right| \leq \frac{1}{\sqrt{\delta_U}} \tag{67}$$

and finally after passing to the limit $\xi_2 \to \xi_1$ we obtain

$$\left|\frac{dy_u(\xi)}{d\xi}\right| \leq \frac{1}{\sqrt{\delta_U}}. \tag{68}$$

From the expression for δ_S we obtain that $\frac{1}{\sqrt{\delta_S}} \leq 67667.96072$. ■

Remark. The estimates in the above lemma may look overgrown, but we need to remember that they are computed on N in coordinates given by c_N ($N \in \{S, U\}$) and will be much smaller when we return to the original coordinates for sets S and U, since the sizes are of order 10^{-4}.

We have

$$z_s(\xi) = z_0 + M_S \cdot (x_s(\xi) \cdot a_S, b_S)^T, \tag{69}$$

$$\frac{\partial}{\partial \xi} z_s(\xi) = M_S \cdot \left(a_s \frac{\partial}{\partial \xi} x_s(\xi), 0 \right)^T \tag{70}$$

and

$$z_u(\xi) = z_0 + M_U \cdot (a_U, y_u(\xi) \cdot b_U)^T, \tag{71}$$

$$\frac{\partial}{\partial \xi} z_u(\xi) = M_U \cdot \left(0, b_U \frac{\partial}{\partial \xi} y_u(\xi) \right)^T; \tag{72}$$

thus, using the estimates from Lemma 9, we get

$$\frac{\partial}{\partial \xi} z_s(\xi) \in ([-0.8359043, 0.8359043], [-0.0504766, 0.0504766])^T, \tag{73}$$

$$\frac{\partial}{\partial \xi} z_u(\xi) \in ([-0.4078735, 0.4078735], [-6.7544925, 6.7544925])^T. \tag{74}$$

We will use these estimates in the proof of Theorem 12.

5.2. Existence of the homoclinic loop. We define the section $L = \{(x, y) \in \mathbb{R}^2 : y = 3\}$, and we denote by L^- and L^+ the lower and upper half-planes separated by L, i.e., $L^- = \{(x, y) : y < 3\}$ and $L^+ = \{(x, y) : y > 3\}$.

Using rigorous integration we show that for $\xi \in \Xi$ there exist

- the forward orbit $\varphi_\xi(t, z_u(\xi))$, $t \in [0, t^+]$, which crosses L at the point $l_u(\xi)$ passing from L^- to L^+ going forward in time and
- the backward orbit $\varphi_\xi(-t, z_s(\xi))$, $t \in [0, t^-]$, which crosses L at the point $l_s(\xi)$ passing from L^+ to L^- going backward in time.

We denote $v_u(\xi) = \pi_x(l_u(\xi))$ and $v_s(\xi) = \pi_x(l_s(\xi))$.

Theorem 10. *For parameter values as in (2) there exists $\xi_h \in \Xi$ such that a homoclinic loop in the system (1) exists.*

Proof. Let U and S be as in Lemmas 7 and 8.

Let us consider the points z_u and z_s from Lemma 8 and their orbits outside the neighborhood of (R_2, Π_2).

From Lemma 8 we know that

- $z_u \in U_l^- = \{z \in U^- : |\pi_x(z)| < R_2\}$, and
- $z_s \in S_l^+ = \{z \in S^+ : |\pi_x(z)| < R_2\}$.

For $\xi \in \Xi$ we define two Poincaré maps:

- $P_u : \Xi \times U_l^- \to L$ such that $P_u(\xi, \cdot)$ maps each point z in U_l^- to $P_u(\xi, z)$—the first point where $\varphi_\xi(\cdot, z)$ crosses L passing from L^- to L^+ going forward in time.
- $P_s : \Xi \times S_l^+ \to L$ such that $P_s(\xi, \cdot)$ maps each point z in S_l^+ to $P_s(\xi, z)$—the first point where $\varphi_\xi(\cdot, z)$ crosses L passing from L^+ to L^- going backward in time.

We see that $l_s(\xi) = P_s(\xi, x_s(\xi))$ and $l_u(\xi) = P_u(\xi, x_u(\xi))$. Using the rigorous computation we were able to prove that P_s and P_u are well defined and we obtain the following estimates:

$$\begin{aligned} v_s(\xi_{lo}) &\in [0.5173639166, 0.5174029455],\\ v_u(\xi_{lo}) &\in [0.5175659545, 0.5175676547],\\ v_s(\xi_{up}) &\in [0.5173641133, 0.5174031422],\\ v_u(\xi_{up}) &\in [0.5166221035, 0.5166237906]. \end{aligned}$$

As we see, $v_s(\xi_{lo}) < v_u(\xi_{lo})$ and $v_u(\xi_{up}) > v_s(\xi_{up})$. Using continuous dependence on the parameter ξ we conclude that there exists a value $\xi_h \in \Xi$ such that $v_s(\xi_h) = v_u(\xi_h)$ and the two curves $\varphi_{\xi_h}(\mathbb{R}, z_s(\xi_h))$ and $\varphi_{\xi_h}(\mathbb{R}, z_u(\xi_h))$ overlap to create the homoclinic loop. ∎

5.3. Local uniqueness of the homoclinic loop. Let $v_s(\xi)$, $v_u(\xi)$, P_s, and P_u be as in the proof of Theorem 10. To prove that the homoclinic orbit is locally unique it suffices to show the following lemma.

Lemma 11. *Let $T : \Xi \to \mathbb{R}$ be a function defined as $T(\xi) = v_s(\xi) - v_u(\xi)$.*

Assume that $\frac{\partial T}{\partial \xi}$ has a constant sign in Ξ not equal to 0.

Then, there exists exactly one parameter value $\xi_h \in [\xi_{lo}, \xi_{up}]$ for which the homoclinic orbit in the system (4) exists.

The proof of Lemma 11 is obvious. More interesting is the fact that we can check the assumptions of the lemma using rigorous numerics.

The derivative of T is given by (see (55))

$$\frac{dT}{d\xi} = \left(\frac{\partial}{\partial \xi}P_s + \frac{\partial P_s}{\partial z}\frac{dz_s}{d\xi}\right) - \left(\frac{\partial}{\partial \xi}P_u + \frac{\partial P_u}{\partial z}\frac{dz_u}{d\xi}\right). \tag{75}$$

Theorem 12. *For parameter values as in (2) there exists a unique $\xi_h \in \Xi$ such that the homoclinic loop in the system (1) exists.*

Proof. The existence of the homoclinic solution was established in Theorem 10. To obtain uniqueness we compute rigorous bounds for (75). As we mentioned earlier, the estimates for partial derivatives of maps P_s and P_u were computed using the CAPD library [CAPD] and the C^1-Lohner algorithm [Z] implemented there.

The estimates of DP_u and DP_s are

$$DP_s \in \big(\, [-0.48196, 0.03188] \quad [-1.37465, 1.34734] \quad [0.09123, 0.16661] \,\big), \tag{76}$$

$$DP_u \in \big(\, [-281.879, 300.027] \quad [-17.0414, 18.1385] \quad [-871.94, -372.028] \,\big), \tag{77}$$

where by $DP_{u,s}$ we understand the matrix

$$DP = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial \xi} \end{pmatrix}.$$

Combining the above bounds with the estimates from Lemma 9 we get the estimate $\frac{dT}{d\xi} \geq 126.7581097 > 0$ which guarantees that T is increasing, and thus the homoclinic loop whose existence we have established earlier is unique due to Lemma 11. ∎

Remark 13. Observe that in (75), since our bounds for $\frac{dz_{u,s}}{\partial \xi}$ are intervals centered at zero, the terms which decide about the sign of $\frac{\partial T}{\partial \xi}$ are $\frac{\partial P_{u,s}}{\partial \xi}$. Therefore we need to have

$$\left|\frac{\partial P_s}{\partial \xi} - \frac{\partial P_u}{\partial \xi}\right| > \left|\frac{\partial P_s}{\partial z}\frac{dz_s}{d\xi}\right| + \left|\frac{\partial P_u}{\partial z}\frac{dz_u}{d\xi}\right|. \tag{78}$$

In our computations we see that the term $\frac{\partial P_u}{\partial \xi}$ dominates the other terms in (75).

Remark 14. The striking feature of the estimates (76) and (77) is the fact that the diameters of the bounds for DP_s are much smaller compared to those for DP_u. Observe that from Figures 1 and 3 it is natural to expect that the computation along the stable manifold should be longer and this should result in worse estimates. The explanation for this is that, in our program, we ran the computations of P_u and P_s with different integration parameters, which gave much more tuning on P_s but at the cost of increased running time. We ran the integration of the stable manifold branch at a fixed time step equal to 0.0001, while for the unstable case we used the automatic step detection provided in CAPD which produced time steps of order $\approx 10^{-2}$. In fact, the computation of P_s took most of the time, while obtaining P_u took only a few seconds.

5.4. Finite time to travel along the homoclinic loop.

Theorem 15. *For parameter values as in* (2) *the travel time along the homoclinic solution of the system* (1) *is finite.*

Proof. Rigorous computation of the bounds of normalized eigenvectors V_u and V_s for the system (4) at the fixed point p_ξ shows that

$$
\begin{aligned}
V_u &\in ([-0.9951981819, -0.9951980492]\,, [0.09788043071, 0.09788178055])^T\,,\\
V_s &\in ([-0.06045616218, -0.06009472876]\,, [0.9981708533, 0.9981926786])^T\,.
\end{aligned}
$$

Now, the finiteness of the time required to travel along W^s and W^u in N for system (1) follows from Theorem 2. Indeed, to have the assumptions of this theorem satisfied it is enough to shift the coordinate origin to (R_2, Π_2) and to change x to $-x$. ■

6. Conclusions.

In this paper we presented geometric tools to establish the bounds on the invariant manifolds and their dependence on the parameter. Those estimates are of quality good enough that the standard method of proving the existence of the homoclinic loop for a fixed point of an ODE could be applied to a planar singular ODE arising from the traveling wave solution ansatz in some hydrodynamic system describing relaxing media.

The tools presented here are not restricted to the case of polynomial equations or to the plane. However, in the nonplanar case, to establish the intersection of the manifolds, which are no longer 1-dimensional, one needs more refined topological tools. An exemplary implementation of such tools in a computer assisted proof can be found in [WZ1].

In the context of the system (1) and also its desingularized version (4) it will be interesting to give a proof of other numerical observations from [V]: the creation of the limit cycle around another fixed point through Andronov–Hopf bifurcation, its growth, and its disappearance. This disappearance happens through collision of the limit cycle with the hyperbolic fixed point, which creates the homoclinic loop studied in the present paper.

Appendix A.

A.1. Derivation of (1).

We will briefly recall from [V, sect. 3] the derivation of the system (1).

Following the papers [DSV, VK] the following system of equations was studied in [V] (equation (9) there) to describe the propagation of an intense pulse in relaxing media:

$$\begin{cases} u_t + p_x = \gamma, \\ V_t - u_x = 0, \\ \tau p_t + \frac{\chi}{V^2} u_x = \frac{\kappa}{V} - p, \end{cases} \tag{79}$$

where u is the mass velocity, V is the specific volume, p is the pressure, γ is the acceleration of the external force, κ and χ/τ are squares of the equilibrium and "frozen" sound velocities, respectively, t is time, and x is the mass (Lagrangian) coordinate.

The first two equations are balance equations for mass and momentum. The third equation is the constitutive relation.

The symmetry analysis in [V] suggests the following ansatz for the traveling wave solution:

$$u = U(\omega), \quad p = \Pi(\omega)(x_0 - x), \quad V = \frac{R(\omega)}{x_0 - x}, \quad \omega = \xi t + \log \frac{x_0}{x_0 - x}.$$

After the above ansatz the second equation in (79) yields

$$U = \xi R + \text{const} \tag{80}$$

and we obtain the system (equation (13) in [V])

$$\begin{cases} \xi\Delta(R)R' = -R(\sigma R\Pi - \kappa + \tau\xi R\gamma), \\ \xi\Delta(R)\Pi' = \xi(\xi R(R\Pi - \kappa) + \chi(\Pi + \gamma)), \end{cases} \tag{81}$$

where $(\cdot)' = \frac{d(\cdot)}{d\omega}$, $\Delta(R) = \tau(\xi R)^2 - \chi$, and $\sigma = 1 + \tau\xi$.

Now, after changing the phase variable $\omega \mapsto \omega/\xi$, we see that the constant ξ on the left-hand side of (81) can be absorbed into the differential operator. After changing the sign of both equations and renaming of variables $(R, \Pi) \mapsto (x, y)$ we obtain the system (1).

A.2. The compacton. We rewrite here the construction of the compacton solution from [V, p. 389]. Assume that we have a homoclinic trajectory to a stationary point (R_2, Π_2). We obtain the compacton-like solution by sewing up the traveling wave solution corresponding to the homoclinic loop with the stationary inhomogeneous solution

$$u = 0, \quad p = \Pi_2(x_0 - x), \quad V = R_2/(x_0 - x), \tag{82}$$

corresponding to the critical point (R_2, Π_2). The reader should notice that strictly speaking we do not have a compactly supported solution here, but if we subtract the stationary inhomogeneous solution given by (82), then we obtain a function with compact support.

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